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THE STEFFENSEN–POPOVICIU MEASURES IN THE CONTEXT OF QUASICONVEX FUNCTIONS

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Abstract. We discuss the extension of Jensen's inequality to the framework of quasiconvex functions and of signed measures.

1. Introduction

Jensen's inequality is an important tool in convex analysis, revealing an essential feature of continuous convex functions under the presence of a mass distribution on their domain. Precisely, if f is a continuous convex function on a compact convex subset K of $\mathbb{R}^N$ and μ is a Borel probability measure on K having the barycenter

$$b_\mu = \int_K x d\mu(x),$$

then the value of f at b_μ does not exceed the mean value of f over K, that is,

$$f(b_\mu) \leqslant \int_K f(x) d\mu(x).$$

Details, various extensions and applications of this results are available in numerous books; see, for example, Niculescu and Persson [18], Pečarić, Proschan and Tong [21], Phelps [22] and Simon [25].

Starting with the pioneering work of J. F. Steffensen [27], Jensen's inequality was extended beyond the framework of positive measures. An account reflecting the state of the art of the early 2000s can be found in [18], Sections 4.1 and 4.2. Part of it is based on [13], [14], [16] and [17]. More recent contributions can be found in [3], [11] and [20].

For the convenience of the reader we will recall here some basic facts.

DEFINITION 1. Let C be a Borel convex subset of $\mathbb{R}^N$. *A* Steffensen-Popoviciu measure on C *is any real Borel measure* μ *on* C *such that* $\mu(C) > 0$ *and*

$$\int_C f(x)\, d\mu(x) \geqslant 0 \quad \textit{for every nonnegative continuous convex function} \quad f : C \to \mathbb{R}.$$

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Clearly, every finite positive measure is also a Steffensen-Popoviciu measure. The following result (due independently to T. Popoviciu [23], and A. M. Fink [2]) gives us a complete characterization of this class of measures in the case where C is a compact interval.

LEMMA 1. *Let μ be a real Borel measure on an interval $[a,b]$ with $\mu([a,b]) > 0$. Then μ is a Steffensen-Popoviciu measure if, and only if, it verifies the following condition of endpoints positivity,*

$$\int_a^t (t-x)\,d\mu(x) \geqslant 0 \text{ and } \int_t^b (x-t)\,d\mu(x) \geqslant 0$$

for every $t \in [a,b]$.

See [18], p. 179, for details.

COROLLARY 1. (Steffensen [27]) *Suppose that $x_1 \leqslant \cdots \leqslant x_n$ are real points and $p_1, \ldots, p_n$ are real weights. Then the discrete measure $\mu = \sum_{k=1}^n p_k \delta_{x_k}$ is a Steffensen-Popoviciu measure if*

$$\sum_{k=1}^n p_k > 0 \quad \text{and} \quad 0 \leqslant \sum_{k=1}^m p_k \leqslant \sum_{k=1}^n p_k \quad \text{for every } m \in \{1, \ldots, n\}. \tag{dSt}$$

Proof. Indeed, according to Lemma 1, the discrete measure $\mu = \sum_{k=1}^n p_k \delta_{x_k}$ is a Steffensen-Popoviciu measure if, and only if,

$$\sum_{k=1}^n p_k > 0, \quad \sum_{k=1}^m p_k(x_m - x_k) \geqslant 0 \quad \text{and} \quad \sum_{k=m}^n p_k(x_k - x_m) \geqslant 0 \tag{dEP}$$

for every $m \in \{1, \ldots, n\}$. Then the fact that (dSt) $\Rightarrow$ (dEP) follows from Abel's summation formula (the discrete analogue of integration by parts). $\square$

Using the integration by parts for absolutely continuous functions (see [5], Corollary 18.20, p. 287), one can prove the following continuous analogue of Corollary 1 above:

COROLLARY 2. *Suppose that $p : [a,b] \to \mathbb{R}$ is a Lebesgue integrable function such that*

$$\int_a^b p(t)dt > 0 \quad \text{and} \quad 0 \leqslant \int_a^x p(t)dt \leqslant \int_a^b p(t)dt \quad \text{for all } x \in [a,b].$$

Then $p(x)dx$ is an absolutely continuous Steffensen-Popoviciu measure on $[a,b]$.

The details are straightforward.

According to Lemma 1, $\left(x^2 + \lambda\right) dx$ is an example of a Steffensen-Popoviciu measure on $[-1,1]$ if, and only if, $\lambda > -1/3$. This measure verifies the conditions of Corollary 2 if and only if $\lambda \geqslant -1/4$.

Several concrete examples of Steffensen-Popoviciu measures illustrating Corollary 2 on $[-1,1]$ are

$$\left(x^2-\frac{1}{6}\right)dx, \quad \left(x^2\pm\frac{x}{2}\right)dx, \quad \left(x^4-\frac{1}{25}\right)dx, \quad \left(x^2-\frac{1}{6}\right)^3 dx.$$

Examples on other intervals:

$$(4x^3-3x)\,dx \quad \text{on} \quad \left[-\frac{1}{2}\sqrt{3},1.5\right]$$

$$\sin x dx \quad \text{on} \quad [0,3\pi]$$

$$\left(1-\frac{1-x}{\pi}-\sin\pi x\right)dx \quad \text{on} \quad [0,1]$$

$$\left[\left(\frac{2x-a-b}{b-a}\right)^2+\lambda\right]dx \quad \text{on } [a,b] \ \ (\text{if} \ \ \lambda \geqslant -1/4)$$

$$\left[\left(\frac{2x-a-b}{b-a}\right)^2-\lambda\frac{2x-a-b}{b-a}\right]dx \quad \text{on} \quad [a,b] \ \ (\text{if} \ \ |\lambda|\leqslant 2/3).$$

The Steffensen-Popoviciu measures on a Borel convex subset of $\mathbb{R}^N$ constitute a convex cone (that includes the cone of finite positive measures). Using the pushing-forward technique of constructing image measures, one can indicate examples of such measures supported by an arbitrarily given compact interval $[a,b]$ (with $a<b$). Notice also that Steffensen-Popoviciu measures can be glued. For example, if $p(x)dx$ and $q(x)dx$ are two such measures on the intervals $[a,c]$ and $[c,b]$ respectively, then

$$\left(p(x)\chi_{[a,c]}+q(x)\chi_{[c,b]}\right)dx$$

is a Steffensen-Popoviciu measure on $[a,b]$. This easily yields examples of Steffensen-Popoviciu measure on $\mathbb{R}$ such as

$$\left[\sum_{n\in\mathbb{Z}}\frac{1}{2^{|n|}}\left((x-2n)^2-\frac{1}{6}\right)\chi_{[2n-1,2n+1]}\right]dx.$$

The connection of Steffensen-Popoviciu measures with Jensen's inequality is as follows:

THEOREM 1. *Suppose that μ is a Steffensen-Popoviciu measure on a compact convex set K. Then $b_\mu=\frac{1}{\mu(K)}\int_K x d\mu(x)\in K$ and for every continuous convex function f on K,*

$$f(b_\mu)\leqslant\frac{1}{\mu(K)}\int_K f(x)\,d\mu(x).$$

For details see [18], Theorem 4.2.1, p. 184.

Usually, Jensen's inequality is viewed as a part of the Hermite-Hadamard double inequality. Some results in the context of signed measures can be found in [3] and [20].

The nice characterization of Steffensen-Popoviciu measures on compact intervals (stated above as Lemma 1) exploits the very simple structure of piecewise linear convex functions in the 1-dimensional case. Indeed, each such function can be represented as

$$\alpha x+\beta+\sum_{k=1}^{m} c_k(x-x_k)^+$$

where $\alpha,\beta\in\mathbb{R}$ and all other coefficients c_k are nonnegative. In higher dimensions $(n\geqslant 2)$ there exist piecewise linear convex functions that cannot be represented as sums of affine functions and positive parts of affine functions. An example is offered by the function

$$\max\{|x|,|y|,2|x+y|-3,2|x-y|-3\},$$

defined on the square $|x|\leqslant 2,\ |y|\leqslant 2$. Needless to say, this situation makes considerably more difficult the study of Steffensen-Popoviciu measures in higher dimensions.

The aim of the present paper is to discuss the usefulness of Steffensen-Popoviciu measures in the framework of quasiconvex functions. These functions appear quite natural in optimal control and differential games, calculus of variations and nonlinear PDEs. In Section 2 we prove that the conditions of Corollary 2 provide the right solution for this extension. See Theorem 2 (and its discrete analogue, Theorem 3). As a consequence we are able to indicate in Section 3 new a priori inequalities for convex functions of higher order. Section 4 calls the attention to the particular case of Rayleigh measures, motivated by the study of Young diagrams and of transition probabilities of the Plancherel measure of the infinite symmetric group.

Section 5 exhibits examples of Steffensen-Popoviciu measures in dimension 2, that are used in the next section to extend Theorem 2 in dimension 2. The paper ends with a list of open questions.

2. The case of quasiconvex functions

Corollary 2 offers a sufficient condition for a signed measure on a compact interval $[a,b]$ to be a Steffensen-Popoviciu measure. We will show that this condition actually assures the positivity of integrals of positive quasiconvex functions.

Recall that a real-valued function f defined on an interval I is called *quasiconvex* if

$$f((1-\lambda)x+\lambda y)\leqslant \max\{f(x),f(y)\}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. The function f is called *quasiconcave* if $-f$ is quasiconvex, that is,

$$f((1-\lambda)x+\lambda y)\geqslant \min\{f(x),f(y)\}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Quasiconvexity is equivalent to the fact that all level sets $L_\lambda=\{x\in I: f(x)\leqslant\lambda\}$ are convex, whenever $\lambda\in\mathbb{R}$. Clearly, every convex function is also quasiconvex, but the converse fails. For example, every monotonic function is quasiconvex. The continuous quasiconvex functions have a nice monotonic behavior, first noticed without proof by S. Johansen [8].

LEMMA 2. *A continuous real-valued function f defined on an interval I is quasiconvex if and only if it is either monotonic or there exists an interior point $c \in I$ such that f is nonincreasing on $(-\infty, c] \cap I$ and nondecreasing on $[c, \infty) \cap I$.*

For details, see the book of Cambini and Martein [1], Theorem 2.5.2, p. 37. This book also contains many valuable examples and applications.

The following result provides the analogue of Lemma 1 in the case of quasiconvex functions and improves Theorem 1 in [11].

THEOREM 2. *Suppose that $g : [a,b] \to \mathbb{R}$ is an integrable function. Then a necessary and sufficient condition in order that*

$$\int_a^b f(x)g(x)dx \geqslant 0$$

for all nonnegative, absolutely continuous and quasiconvex functions $f : [a,b] \to \mathbb{R}$ is that

$$\int_a^x g(t)dt \geqslant 0 \text{ and } \int_x^b g(t)dt \geqslant 0 \quad \text{for every } x \in [a,b]. \tag{St}$$

Proof. The Sufficiency. According to Lemma 2, there exists a point $c \in [a,b]$ such that f is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. Then

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= \int_a^c f(x)g(x)dx + \int_c^b f(x)g(x)dx \\
&= \int_a^c f(x)d\left(\int_a^x g(t)dt\right) - \int_c^b f(x)d\left(\int_x^b g(t)dt\right) \\
&= \left[f(x)\int_a^x g(t)dt\right]\Big|_a^c - \int_a^c f'(x)\left(\int_a^x g(t)dt\right)dx \\
&\quad - \left[f(x)\int_x^b g(t)dt\right]\Big|_c^b + \int_c^b f'(x)\left(\int_x^b g(t)dt\right)dx \\
&= f(c)\int_a^c g(t)dt + \int_a^c (-f'(x))\left(\int_a^x g(t)dt\right)dx \\
&\quad + f(c)\int_c^b g(t)dt + \int_c^b f'(x)\left(\int_x^b g(t)dt\right)dx \geqslant 0,
\end{aligned}$$

as a sum of nonnegative numbers. The integration by parts for absolutely continuous functions is motivated by Theorem 18.19, p. 287, in the monograph of Hewitt and Stromberg [5].

The Necessity. Assuming $x_0 \in (a,b)$ and $\varepsilon > 0$ sufficiently small, the function

$$L_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in [a, x_0 - \varepsilon] \\ -(x - x_0)/\varepsilon & \text{if } x \in [x_0 - \varepsilon, x_0] \\ 0 & \text{if } x \in [x_0, b] \end{cases}$$

is nonnegative, decreasing and also absolutely continuous. Therefore

$$\int_a^{x_0} g(t)dt = \lim_{\varepsilon\to 0}\int_a^b L_\varepsilon(x)g(x)dx \geqslant 0.$$

In a similar way one can prove that $\int_{x_0}^b g(t)dt \geqslant 0$. □

Since every continuous convex function $f:[a,b]\to\mathbb{R}$ is absolutely continuous (see [18], Proposition 1.6.1, p. 37), Theorem 2 implies Corollary 2 above.

It is worth noticing that the discrete measures $\sum_{k=1}^n p_k\delta_{x_k}$ described in Corollary 1 provide the discrete analogue of Theorem 2:

THEOREM 3. *Suppose that $x_1\leqslant\cdots\leqslant x_n$ are real points and $p_1,\ldots,p_n$ are real weights. Then the discrete measure $\sum_{k=1}^n p_k\delta_{x_k}$ has the property that*

$$\sum_{k=1}^n p_k f(x_k)\geqslant 0$$

for every nonnegative continuous quasiconvex function $f:[a,b]\to\mathbb{R}$ if and only if

$$0\leqslant\sum_{k=1}^m p_k\leqslant\sum_{k=1}^n p_k \quad \textit{for every } m\in\{1,\ldots,n\}.$$

The proof is similar to that of Theorem 2 and we omit the details.

REMARK 1. The signed measures of the form $g(x)dx$ with g verifying the conditions (St) are not monotonic in general. For example, the integral of the nonnegative quasiconcave function

$$h(x)=\begin{cases} 0 & \text{if } x\in[-1,-1/\sqrt{6}] \\ x\sqrt{6}+1 & \text{if } x\in[-1/\sqrt{6},0] \\ -x\sqrt{6}+1 & \text{if } x\in[0,1/\sqrt{6}] \\ 0 & \text{if } x\in[1/\sqrt{6},1] \end{cases}$$

with respect to the measure $\left(x^2-\frac{1}{6}\right)dx$ is negative. Theorem 2 shows that the property of monotonicity takes place on the lattice generated by the nonnegative convex functions. As concerns the nonnegative concave functions f, this theorem yields a much weaker property, precisely,

$$\int_a^b f(x)g(x)dx\leqslant\max_{a\leqslant x\leqslant b} f(x)\int_a^b g(x)dx.$$

To see this, apply Theorem 2 to the function $-f(x)+\max_{a\leqslant x\leqslant b}f(x)$ (that is convex and nonnegative).

3. Inequalities of convexity

The following result shows how to associate to Steffensen-Popoviciu measures a priori inequalities for higher order convex functions (for example, for those convex functions which are also 3-convex in the sense of Popoviciu [23], [21]). It illustrates the case of the measure

$$\left[\left(\frac{2x-a-b}{b-a}\right)^2-\frac{1}{4}\right]dx$$

on the interval $[a,b]$ and follows from the corrected trapezoidal rule (discussed by Talvila and Wiersma in [28]).

THEOREM 4. *Suppose that $u:[a,b]\to\mathbb{R}$ is a twice differentiable convex function whose second derivative is an absolutely continuous quasiconvex function. Then*

$$\frac{1}{b-a}\int_a^b u(x)dx \geqslant \frac{u(a)+u(b)}{2}-\frac{3(b-a)}{32}\left(u'(b)-u'(a)\right).$$

Proof. Indeed,

$$\begin{aligned}
0 &\leqslant \int_a^b u''(x)\left[\left(\frac{2x-a-b}{b-a}\right)^2-\frac{1}{4}\right]dx \\
&= \left[\left(\frac{2x-a-b}{b-a}\right)^2-\frac{1}{4}\right]u'(x)\Big|_a^b - \frac{4}{b-a}\int_a^b \frac{2x-a-b}{b-a}u'(x)dx \\
&= \frac{3}{4}\left(u'(b)-u'(a)\right)-\frac{4}{b-a}\,\frac{2x-a-b}{b-a}u(x)\Big|_a^b+\frac{8}{(b-a)^2}\int_a^b u(x)dx \\
&= \frac{3}{4}\left(u'(b)-u'(a)\right)-\frac{4}{b-a}\left(u(b)+u(a)\right)+\frac{8}{(b-a)^2}\int_a^b u(x)dx. \quad \square
\end{aligned}$$

According to Hammer's variant of the Hermite–Hadamard inequality (see [4] or [18], Remark 1.9.3, p. 52), the mean value of a convex function u verifies the upper estimate

$$\frac{1}{b-a}\int_a^b u(x)dx \leqslant \frac{1}{2}\left[\frac{u(a)+u(b)}{2}+u\left(\frac{a+b}{2}\right)\right],$$

so under the assumptions of Theorem 4 we infer that

$$\frac{3(b-a)}{16}\left(u'(b)-u'(a)\right)+u\left(\frac{a+b}{2}\right)\geqslant\frac{u(a)+u(b)}{2}. \tag{$*$}$$

The hypotheses of Theorem 4 are fulfilled by any *completely monotonic function*, that is, by any infinitely differentiable function $f:(0,\infty)\to\mathbb{R}$ such that

$$(-1)^n f^{(n)}(x)\geqslant 0 \text{ for all } x>0 \text{ and } n\in\mathbb{N}.$$

This is a very rich class of functions. See the survey of Miller and Samko [12]. In the particular case where $u(x) = -\log x$, the inequality $(*)$ becomes

$$\exp \frac{3(b-a)^2}{16ab} \geqslant \frac{\frac{a+b}{2}}{\sqrt{ab}} \quad \text{for } b \geqslant a > 0,$$

while for $u(x) = x\log x$ it implies the inequality

$$\left(\frac{b}{a}\right)^{\frac{3(b-a)}{16}} \left(\frac{a+b}{2}\right)^{\left(\frac{a+b}{2}\right)} \geqslant \left(a^{a/2}b^{b/2}\right) \quad \text{for } b \geqslant a > 0.$$

REMARK 2. It was noticed in [15] that the restriction of the signed measure

$$\left[\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{6}\right] dx$$

to the interval $[a,b]$ implies inequalities of the form

$$\int_a^b u(x) \left[\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{6}\right] dx \geqslant 0,$$

for all nonnegative, continuous and concave functions $f : [a,b] \to \mathbb{R}$. Using the argument of Theorem 4, one can show that

$$\frac{1}{b-a}\int_a^b u(x)dx \geqslant \frac{u(a)+u(b)}{2} - \frac{5(b-a)}{48}\left(u'(b) - u'(a)\right)$$

for every twice differentiable convex function $u : [a,b] \to \mathbb{R}$ whose second derivative is continuous and concave. An example of such function is the restriction of $1 - \sin x$ on $[0,\pi]$.

4. Rayleigh measures

The notion of Rayleigh measure was introduced by Kerov [9] in a paper dedicated to the relationship between probability distributions μ and certain bounded signed measures τ on the real line $\mathbb{R}$, satisfying the *Markov-Krein identity*,

$$\int_{-\infty}^{\infty} \frac{d\mu(u)}{z-u} = \exp \int_{-\infty}^{\infty} \ln \frac{1}{z-u} d\tau(u) \quad \text{for } z \in \mathbb{C},\ \operatorname{Im} z > 0. \tag{MK}$$

Precisely, the function $R(z)$ denoting the right hand side can be represented as the Cauchy-Stieltjes transform of a probability measure μ,

$$R(z) = \int_{-\infty}^{\infty} \frac{d\mu(u)}{z-u},$$

whenever τ is a *Rayleigh measure*, that is, a bounded signed measure that verifies the following three properties:

$$0 \leqslant \tau(\{x : x < a\}) \leqslant 1 \quad \text{for every } a \in \mathbb{R}; \tag{RM1}$$

$$\tau((-\infty, \infty)) = 1; \tag{RM2}$$

$$\int_{-\infty}^{\infty} \ln(1 + |x|)\, d|\tau|(x) < \infty. \tag{RM3}$$

See [9], Corollary 2.4.1.

An alternative approach of Rayleigh measures is offered by the notion of interlace measures. Two finite positive measures τ' and τ'' *interlace* if there exists a Rayleigh measure τ such that $\tau' = \tau^+$ and $\tau'' = \tau^-$. A simple example is provided by Cauchy's interlace theorem: Let A be an $n \times n$ dimensional Hermitian matrix and B its principal $(n-1) \times (n-1)$ submatrix obtained by deleting the last row and column. If $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ lists the eigenvalues of A and $\mu_1 \leqslant \cdots \leqslant \mu_{n-1}$ of B, then

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n. \tag{C}$$

A simple proof (based on the Courant-Fischer min-max theorem) can be found in Horn and Johnson [6], Theorem 4.3.17.

Notice that under the above circumstances, the discrete measure $\tau = \sum_{k=1}^{n} \delta_{\lambda_k} - \sum_{j=1}^{n-1} \delta_{\mu_j}$ is an example of Rayleigh measure and the corresponding Markov-Krein identity has the form

$$\sum_{k=1}^{n} \frac{p_k}{z - \lambda_k} = \frac{(z-\mu_1)\cdots(z-\mu_{n-1})}{(z-\lambda_1)\cdots(z-\lambda_n)}$$

for suitable $p_1, \ldots, p_n > 0$ with $\sum_{k=1}^{n} p_k = 1$. According to Theorem 3, the interlace sequences (C) lead to the following result:

PROPOSITION 1. *We have*

$$\sum_{k=1}^{n-1} f(\mu_k) + \min_{x \in [\lambda_1, \lambda_n]} f(x) \leqslant \sum_{k=1}^{n} f(\lambda_k),$$

for every continuous quasiconvex function f defined on an interval containing the eigenvalues of A.

An important source of interlacing sequences is provided by the orthogonal polynomials. If $(P_n(x))_n$ is a sequence of polynomials orthogonal with respect to a positive measure, then the roots of any two consecutive polynomials $P_n(x)$ and $P_{n-1}(x)$ interlace. So are the classical sequences of Legendre polynomials and of Chebyshev polynomials.

It is worth noticing that Proposition 1 also works in the context of orthogonal polynomials. The Legendre polynomials $P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$ are orthogonal on the interval $[-1, 1]$ with respect to the Lebesgue measure dx. If $x_1 \leqslant \cdots \leqslant x_n$ are the roots of $P_n(x)$ and $y_1 \leqslant \cdots \leqslant y_{n-1}$ of $P_{n-1}(x)$, then

$$\sum_{k=1}^{n-1} f(y_k) + \min_{x \in [-1,1]} f(x) \leqslant \sum_{k=1}^{n} f(x_k),$$

for every continuous quasiconvex function $f : [-1,1] \to \mathbb{R}$. This remark shed new light on the Bruijn-Springer-Malamud inequality (as proved in [10]).

Interesting examples of absolutely continuous Rayleigh measures can be found in the paper of Romik [24].

5. Simple examples of Steffensen-Popoviciu measures in dimension 2

EXAMPLE 1. If $p(x)dx$ and $q(y)dy$ are Steffensen-Popoviciu measures on the intervals $[a,b]$ and respectively $[c,d]$, then $(p(x)+q(y))\,dxdy$ is a measure of the same type on $[a,b]\times[c,d]$. In fact, if $f : [a,b]\times[c,d] \to \mathbb{R}$ is a nonnegative convex function, then

$$x \to \left(\int_c^d f(x,y)dy\right) \text{ and } y \to \left(\int_a^b f(x,y)dx\right)$$

are also nonnegative convex functions and thus

$$\begin{aligned}&\int_a^b \int_c^d f(x,y)\,(p(x)+q(y))\,dydx \\ &= \int_a^b \left(\int_c^d f(x,y)dy\right) p(x)dx \int_c^d \left(\int_a^b f(x,y)dx\right) q(y)dy \geqslant 0\end{aligned}$$

as a sum of nonnegative numbers. In particular, $(\lambda p(x)+\mu q(y))\,dxdy$ is a Steffensen-Popoviciu measure on $[a,b]\times[c,d]$ whenever $\lambda,\mu \geqslant 0$.

Using appropriate affine changes of variables, one can construct Steffensen- Popoviciu measures on every quadrilateral convex domain. Here is worth noticing that the superposition of a convex function and an affine function is also a convex function.

EXAMPLE 2. Suppose that $p(x)dx$ and $q(y)dy$ are Steffensen-Popoviciu measures on the intervals $[a,b]$ and respectively $[c,d]$. The argument above shows that $p(x)q(y)dxdy$ is a Steffensen-Popoviciu measure on $[a,b]\times[c,d]$ if in addition $p(x)$ or $q(y)$ is a nonnegative function.

This type of measures also works in the case of triangular domains of the form $\{(x,y) : x \geqslant 0,\ y \geqslant 0,\ x+y \leqslant C\}$ for $C > 0$. Therefore, using appropriate affine changes of variables, one can construct Steffensen-Popoviciu measures on every triangular domain.

REMARK 3. If $p(x)dx$ and $q(y)dy$ are two Steffensen-Popoviciu measures as in the preceding example and $f : [a-d, b-c] \to \mathbb{R}$ is a nonnegative convex function of class C^2 whose second derivative is also convex, then

$$\int_a^b \int_c^d f(x-y)p(x)q(y)dydx \geqslant 0.$$

Indeed, assuming that $q \geqslant 0$ (to make a choice), we infer that the function $x \to \int_c^d f(x-y)q(y)dy$ is nonnegative and convex.

EXAMPLE 3. If $p(x)dx$ is a Steffensen-Popoviciu measure on the interval $[0,R]$, then $\frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}dxdy$ is a Steffensen-Popoviciu measure on the compact disc

$$\overline{D}_R(0) = \left\{(x,y) : x^2 + y^2 \leqslant R\right\}.$$

This follows from the usual formula of converting from rectangular to polar coordinates.

In particular, $\frac{\left(2\sqrt{x^2+y^2}-1\right)^2-\frac{1}{4}}{\sqrt{x^2+y^2}}dxdy$ is a Steffensen-Popoviciu measure on the compact unit disc as well as on each sector of it.

6. Extension of Theorem 2

Some of the results noticed above have 2-dimensional analogues. In particular, this is the case of Theorem 2.

THEOREM 5. *Suppose that $p : [a,b] \to \mathbb{R}$ and $q : [c,d] \to \mathbb{R}$ are continuous functions such that at least one of the following two conditions are fulfilled:*
(i) $p \geqslant 0$ and $0 \leqslant \int_c^y q(s)ds \leqslant \int_c^d q(s)ds$ for all $y \in [c,d]$;
(ii) $q \geqslant 0$ and $0 \leqslant \int_a^x p(t)dt \leqslant \int_a^b p(t)dt$ for all $x \in [a,b]$.
Then

$$\int_a^b \int_c^d f(x,y)p(x)q(y)dydx \geqslant 0$$

for every nonnegative, continuously differentiable and quasiconvex function $f : [a,b] \times [c,d] \to \mathbb{R}$.

The condition of continuous differentiability on f can be replaced by that of absolute continuity in the sense of Carathéodory. See Šremr [26] for details.

Proof. (i). We start by choosing a continuous path $x \to c^*(x)$ such that

$$f(x,c^*(x)) = \min\{f(x,y) : c \leqslant y \leqslant d\} \quad \text{for each} \quad x \in [a,b].$$

Then $y \to f(x,y)$ is nonincreasing on $[c,c^*(x)]$ and nondecreasing on $[c^*(x),d]$, which implies

$$\frac{\partial f}{\partial y}(x,y) \leqslant 0 \quad \text{on} \quad [c,c^*(x)] \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) \geqslant 0 \quad \text{on} \quad [c^*(x),d].$$

Therefore

$$
\begin{aligned}
&\int_a^b \int_c^d f(x,y)p(x)q(y)dydx \\
&= \int_a^b \left[\int_c^{c^*(x)} f(x,y)d\left(\int_c^y q(t)dt\right)\right] p(x)dx \\
&\quad - \int_a^b \left[\int_{c^*(x)}^d f(x,y)d\left(\int_y^d q(t)dt\right)\right] p(x)dx \\
&= \int_a^b \left[f(x,y)\int_c^y q(t)dt\Big|_c^{c^*(x)} - \int_c^{c^*(x)} \left(\frac{\partial f}{\partial y}(x,y)\int_c^y q(t)dt\right)dy\right] p(x)dx \\
&\quad - \int_a^b \left[f(x,y)\int_y^d q(t)dt\Big|_{c^*(x)}^d - \int_{c^*(x)}^d \left(\frac{\partial f}{\partial y}(x,y)\int_y^d q(t)dt\right)dy\right] p(x)dx \\
&= \int_a^b f(x,c^*(x))\left(\int_c^{c^*(x)} q(t)dt\right)p(x)dx \\
&\quad + \int_a^b \left[\int_c^{c^*(x)} \left(\frac{-\partial f}{\partial y}(x,y)\int_c^y q(t)dt\right)dy\right] p(x)dx \\
&\quad + \int_a^b f(x,c^*(x))\left(\int_{c^*(x)}^d q(t)dt\right)p(x)dx \\
&\quad + \int_a^b \left[\int_{c^*(x)}^d \left(\frac{\partial f}{\partial y}(x,y)\int_y^d q(t)dt\right)dy\right] p(x)dx \\
&= \int_c^d q(t)dt \int_a^b f(x,c^*(x))p(x)dx + \int_a^b \left[\int_c^{c^*(x)} \left(\frac{-\partial f}{\partial y}(x,y)\int_c^y q(t)dt\right)dy\right] p(x)dx \\
&\quad + \int_a^b \int_{c^*(x)}^d \left[\left(\frac{\partial f}{\partial y}(x,y)\int_y^d q(t)dt\right)dy\right] p(x)dx \geqslant 0
\end{aligned}
$$

as a sum of nonnegative numbers. The proof is done. $\square$

Theorem 5 works for

$$p(x) = 1 \text{ and } q(y) = \left(\frac{2y-c-d}{d-c}\right)^2 - \frac{1}{4},$$

as well as for

$$p(x) = \left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{4} \text{ and } q(y) = 1.$$

Combining these two cases we infer that

$$\int_a^b \int_c^d f(x,y)\left(\left(\frac{2x-a-b}{b-a}\right)^2 + \left(\frac{2y-c-d}{d-c}\right)^2 - \frac{1}{2}\right)dydx \geqslant 0$$

for every nonnegative, continuously differentiable and quasiconvex function $f : [a,b] \times [c,d] \to \mathbb{R}$.

Some simple examples of nonnegative quasiconvex functions to which this remark applies are

$$\begin{aligned}
&f(x,y) = (x\log x)/y \quad \text{for} \quad x \geqslant 1,\ y > 0\\
&f(x,y) = 1/\sqrt{xy} \quad \text{for} \quad x,y > 0\\
&f(x,y) = (ax^{\alpha} + by^{\alpha})^{1/\alpha} \quad \text{for} \quad x,y \geqslant 0 \quad (a,b > 0,\ \alpha > 0).
\end{aligned}$$

REMARK 4. In the case of the compact disc $\overline{D}_R(0)$ one can prove the following partial analogue of Theorem 2: If $p(x)dx$ is a Steffensen-Popoviciu measure on the interval $[0,R]$, then

$$\iint_{\overline{D}_R(0)} f(x,y)\frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}dxdy \geqslant 0$$

for every nonnegative, continuously differentiable and quasiconvex function $f:\overline{D}_R(0) \to \mathbb{R}$. This follows by passing to polar coordinates and taking into account Theorem 2.

7. Open problems

In Section 1 we mentioned the existence of Steffensen-Popoviciu measures having $\mathbb{R}$ as support, but a characterization comparable to Lemma 1 for such measures is unknown. We also lack an analogue of Lemma 1 in dimension 2 (or higher).

According to [15], a real Borel measure μ on an interval I is said to be a *dual Steffensen-Popoviciu measure* if $\mu(I) > 0$ and

$$\int_I f(x)\,d\mu(x) \geqslant 0$$

for every nonnegative continuous concave function $f: I \to \mathbb{R}$. A consequence of the fact that $\left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{6}\right)dx$ is a dual Steffensen-Popoviciu measure on $I=[a,b]$ made the object of Remark 2 above.

At this moment no characterization of dual Steffensen-Popoviciu measures is available.

As was noticed by Malamud [10], Cauchy's interlace theorem can be extended to the case of normal measures. It seems very likely that this fact yields Steffensen-Popoviciu discrete measures on the complex plane (but we lack a formal proof).

Last but not least, the possible applications of Steffensen-Popoviciu measures to optimal control, game theory, variational calculus etc need further investigation.

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